A NOTE ON THE INVARIANCE IN THE NONABELIAN TENSOR PRODUCT

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Abstract. In the nonabelian tensor product $G \otimes H$ of two groups G and H many properties pass from G and H to $G \otimes H$. There is a wide literature for different properties involved in this passage. We look at weak conditions for which such a passage may happen.

1. Terminology and statement of the result

Let G and H be two groups acting upon each other in a *compatible way*:

$$ {}^{{}^g h}g' = {}^{g}({}^{h}({}^{g^{-1}}h')), \qquad {}^{{}^h g}h' = {}^{h}({}^{g}({}^{h^{-1}}h')), \tag{1.1}$$

for $g, g' \in G$ and $h, h' \in H$, and acting upon themselves by conjugation. The *nonabelian tensor product* $G \otimes H$ of G and H is the group generated by the symbols $g \otimes h$ with defining relations

$$ gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h), \qquad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'). \tag{1.2}$$

When $G = H$ and all actions are by conjugations, $G \otimes G$ is called *nonabelian tensor square* of G. These notions were introduced in [3, 4] and some significant contributions can be found in [1, 2, 5, 6, 8, 9, 10, 12, 13].

From the defining relations in $G \otimes H$,

$$ \kappa : g \otimes h \in G \otimes H \mapsto \kappa(g \otimes h) = [g, h] \in [G, H] = \langle g^{-1}h^{-1}gh \mid g \in G, h \in H\rangle \tag{1.3}$$

is an epimorphism of groups. Still from [3, 4], if G and H act trivially upon each other, then $G \otimes H$ is isomorphic to the usual tensor product $G^{ab} \otimes_{\mathbb{Z}} H^{ab}$. If they act compatibly upon each other, then their actions induce an action of the free product $G * H$ on $G \otimes H$ given by ${}^{x}(g \otimes h) = {}^{x}g \otimes {}^{x}h$, where $x \in G * H$.

The *exterior product* $G \wedge H$ is the group obtained with the additional relation $g \otimes h = 1_{\otimes}$ on $G \otimes H$, that is,

$$ G \wedge H = (G \otimes H)/D, \tag{1.4}$$

where $D = \langle g \otimes g : g \in G \cap H\rangle$. Now it is easy to check that

$$ \kappa' : g \wedge h \in G \wedge H \mapsto \kappa'(g \wedge h) = [g, h] \in [G, H] \tag{1.5}$$

is a well–defined epimorphism of groups. For convenience of the reader, we recall that there is a famous commutative diagram with exact rows and central extensions as columns in [3, (1)]: It correlates the second homology group $H_2(G)$ of G with the third homology group $H_3(G)$ of G, the Whitehead's quadratic functor Γ, the Whitehead's function ψ and $\ker \kappa = J_2(G)$ (see also [3, 4, 14]).

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Now we get to the purpose of the present paper. Given a class of groups $\mathfrak{X}$, many authors answered the question:

$$\text{If} \quad G, H \in \mathfrak{X}, \quad \text{then} \quad G \otimes H \in \mathfrak{X} \tag{1.6}$$

In case $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, see [5]. In case $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, see [2, 13]. In case $\mathfrak{X} = \mathfrak{S}$ is the class of all soluble groups, see [10, 13]. In case $\mathfrak{X} = \mathfrak{P}$ is the class of all polycyclic groups, see [8]. In case $\mathfrak{X} = \mathbf{L}\mathfrak{F}$ is the class of all locally finite groups, see [9]. In case $\mathfrak{X} = \check{\mathfrak{C}}$ (resp., $\mathfrak{X} = \mathfrak{S}_2$) is the class of all Chernikov (resp., soluble minimax) groups, see [11]. Some topological properties are also closed with respect to forming the nonabelian tensor product, as observed in [3, 4].

We recall some notations from [7].

- $\mathfrak{X} = \mathbf{S}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming subgroups.
- $\mathfrak{X} = \mathbf{H}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming homomorphic images.
- $\mathfrak{X} = \mathbf{P}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming extensions, i.e.: if $N \in \mathfrak{X}$ is a normal subgroup of G and $G/N \in \mathfrak{X}$, then $G \in \mathfrak{X}$.
- $\mathfrak{X} = \mathbf{H_2}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming the second homology group, i.e.: if $G \in \mathfrak{X}$, then $H_2(G) \in \mathfrak{X}$.
- $\mathfrak{X} = \mathbf{H_3}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming the third homology group, i.e.: if $G \in \mathfrak{X}$, then $H_3(G) \in \mathfrak{X}$.
- $\mathfrak{X} = \mathbf{T}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming (usual) abelian tensor products , i.e.: if $A, B \in \mathfrak{X}$ are abelian, then $A \otimes_{\mathbb{Z}} B \in \mathfrak{X}$.

Our main contribution is below.

Main Theorem. *Let G and H be two groups, acting compatibly upon each other and $\mathfrak{X} = \mathbf{S}\mathfrak{X} = \mathbf{H}\mathfrak{X} = \mathbf{P}\mathfrak{X} = \mathbf{H_2}\mathfrak{X} = \mathbf{H_3}\mathfrak{X} = \mathbf{T}\mathfrak{X}$. If $G, H, \Gamma((G \cap H)^{ab}) \in \mathfrak{X}$, then $G \otimes H \in \mathfrak{X}$.*

In [2, 5, 8, 9, 10, 11, 13], the quoted results follow from Main Theorem, when we choose $\mathfrak{X}$ among $\mathfrak{F}, \mathfrak{N}, \mathfrak{S}, \mathfrak{P}, \mathbf{L}\mathfrak{F}, \check{\mathfrak{C}}, \mathfrak{S}_2$.

2. Proof and some consequences

We illustrate that it is possible to adapt an argument in [8, Section 2].

Proof of Main Theorem. Let $P = G * H/IJ$ be the Pfeiffer product of G and H, where I and J are the normal closures in $G * H$ of $\langle {}^h g h g^{-1} h^{-1} : g \in G, h \in H\rangle$ and $\langle {}^g h g h^{-1} g^{-1} : g \in G, h \in H\rangle$, respectively. See [8, 14]. Note that P is a homomorphic image of $G \ltimes H$, hence $P \in \mathfrak{X}$. Here we have used $\mathfrak{X} = \mathbf{H}\mathfrak{X}$. Let $\mu : G \to P$ and $\nu : H \to P$ be inclusions. Denote $\overline{G} = \mu(G)$ and $\overline{H} = \nu(H)$. Then $\overline{G}$ and $\overline{H}$ are normal subgroups of P and $P = \overline{G}\,\overline{H}$. Of course, $\ker \mu \leq Z(G)$ and $\ker \nu \leq Z(H)$. An argument as in [3, Proposition 9] shows that the following sequence is exact:

$$(G \otimes \ker \nu) \times (\ker \mu \otimes H) \xrightarrow{i} G \otimes H \longrightarrow \overline{G} \otimes \overline{H} \longrightarrow 1, \tag{2.1}$$

where i is the inclusion $(g \otimes h', g' \otimes h) \mapsto (g \otimes h')(g' \otimes h)$. It is easy to see that $\mathrm{Im}\ i \leq Z(G \otimes H)$. Since ${}^h g =^{\nu(g)} g$ and ${}^g h =^{\mu(g)} h$, $\ker \mu$ and $\ker \nu$ act trivially on H and G, respectively.

Therefore,

$$G \otimes \ker \nu \simeq G^{ab} \otimes_{\mathbb{Z}} \ker \nu^{ab} = G^{ab} \otimes_{\mathbb{Z}} \ker \nu \tag{2.2}$$

and

$$\ker \mu \otimes H \simeq \ker \mu^{ab} \otimes_{\mathbb{Z}} H^{ab} = \ker \mu \otimes_{\mathbb{Z}} H^{ab}. \tag{2.3}$$

In particular, $G \otimes \ker \nu \in \mathfrak{X}$. Here we have used $\mathfrak{X} = \mathbf{T}\mathfrak{X}$. Analogously, $\ker \mu \otimes H \in \mathfrak{X}$. It follows that $\mathrm{Im}\ i \in \mathfrak{X}$ because it is a homomorphic image of $(G \otimes \ker \nu) \times (\ker \mu \otimes H) \in \mathfrak{X}$. Still we have used $\mathfrak{X} = \mathbf{H}\mathfrak{X}$.

Since $\overline{G} \otimes \overline{H} \simeq (G \otimes H)/\mathrm{Im}\ i$, it is enough to prove that $\overline{G} \otimes \overline{H} \in \mathfrak{X}$. Here we have used $\mathfrak{X} = \mathbf{P}\mathfrak{X}$. We may work with $\overline{G}$ instead of G and with $\overline{H}$ instead of H in order to get our result. Then there is no loss of generality in assuming that G and H are normal subgroups of P, $P = GH$, and all actions are induced by conjugation in P. Note that $(G \wedge H)/\ker \kappa'$ is isomorphic to $[G, H] \leq G \cap H \leq G \in \mathfrak{X}$ and so $(G \wedge H)/\ker \kappa' \in \mathfrak{X}$. Here we have used $\mathfrak{X} = \mathbf{H}\mathfrak{X} = \mathbf{S}\mathfrak{X}$. If we prove $\ker \kappa' \in \mathfrak{X}$, then $G \wedge H \in \mathfrak{X}$ by $\mathfrak{X} = \mathbf{P}\mathfrak{X}$. If we prove also $D \in \mathfrak{X}$, then $G \otimes H \in \mathfrak{X}$, still by $\mathfrak{X} = \mathbf{P}\mathfrak{X}$ and we are done.

By [4, Theorem 4.5], we have an exact sequence:

$$\longrightarrow H_3(P/G) \oplus H_3(P/H) \longrightarrow \ker \kappa' \longrightarrow H_2(P) \longrightarrow . \tag{2.4}$$

Since $P, P/G, P/H \in \mathfrak{X}$, we have $H_2(P), H_3(P/G), H_3(P/H) \in \mathfrak{X}$. Here we have used $\mathfrak{X} = \mathbf{H}\mathfrak{X} = \mathbf{H_2}\mathfrak{X} = \mathbf{H_3}\mathfrak{X}$. On the other hand, $\ker \kappa'$ is an extension of $H_3(G/M) \oplus H_3(G/N) \in \mathfrak{X}$ by $H_2(G) \in \mathfrak{X}$. Therefore, $\ker \kappa' \in \mathfrak{X}$, as claimed. Here we have used $\mathfrak{X} = \mathbf{P}\mathfrak{X}$.

Having in mind the famous diagram [3, (1)], it is easy to check that there exists a well–defined homomorphism of groups $\psi : \Gamma((G \cap H)^{ab}) \to (G \cap H) \otimes (G \cap H)$. See [3, p.181] or [4]. Then $\mathrm{Im}\ \psi = D \in \mathfrak{X}$, as claimed. Here we have used $\mathfrak{X} = \mathbf{H}\mathfrak{X}$ and $\Gamma((G \cap H)^{ab}) \in \mathfrak{X}$.

The result follows.

□

Note that $\Gamma(G^{ab})$ plays a fundamental role in deciding if $G \otimes G \in \mathfrak{X}$. This was already noted in [2, Section 3] for the class of all free nilpotent groups of finite rank. Then it is clear that the following corollary extends many results in [2, 5, 8, 9, 10, 11, 13] in case of the nonabelian tensor square.

Corollary.
Assume $G = H$ in Main Theorem. If $G, \Gamma(G^{ab}) \in \mathfrak{X}$, then $G \otimes G \in \mathfrak{X}$.

We end with two observations on the invariance with respect to the nonabelian tensor product.

Remark 1. Sometimes it is enough that $[G, H] \in \mathfrak{X}$ in order to decide whether $G \otimes H \in \mathfrak{X}$. In case of $\mathfrak{X} = \mathfrak{N}$, or $\mathfrak{X} = \mathfrak{S}$, this can be found in [3, 10, 13].

The second deals with the *universal property* of the nonabelian tensor products.

Remark 2. In a certain sense the *universal property of the nonabelian tensor products* (see [4]) justifies Main Theorem, because it shows that we need at least $\mathfrak{X} = \mathbf{S}\mathfrak{X} = \mathbf{H}\mathfrak{X} = \mathbf{P}\mathfrak{X}$, if we hope to answer (1.6) positively.

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